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The modular automorphism group of a Poisson manifold

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Abstract

The modular flow of Poisson manifold is a 1-parameter group of automorphisms determined by the choice of a smooth density on the manifold. When the density is changed, the generator of the group changes by a hamiltonian vector field, so one has a 1-parameter group of "outer automorphisms" intrinsically attached to any Poisson manifold. The group is trivial if and only if the manifold admits a measure which is invariant under all hamiltonian flows.

The notion of modular flow in Poisson geometry is a classical limit of the notion of modular automorphism group in the theory of von Neumann algebras. In addition, the modular flow of a Poisson manifold is related to modular cohomology classes for associated Lie algebroids and symplectic groupoids. These objects have recently turned out to be important in Poincaré duality theory for Lie algebroids.

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Dedicated to Professor A. Lichnerowicz

1. Introduction

The modular automorphism group of a von Neumann algebra A is a 1-parameter group of automorphisms of A which, modulo inner automorphisms, is canonically associated to A. Since Poisson manifolds can be thought of as "semiclassical limits" of operator algebras, it is natural to ask whether they, too, have modular automorphism groups. This paper will show that they do. Thus, in the terms of Connes [4], Poisson manifolds are, like von Neumann algebras, intrinsically *dynamical* objects. It appears that the study of the modular vector

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field will give some geometric insight into the modular automorphisms of von Neumann algebras, as well as being a useful tool in Poisson geometry.

The modular theory of von Neumann algebras has its origins in the KMS theory in quantum mechanics and the closely related Tomita–Takesaki theory. In these theories, a 1-parameter group of automorphisms is related to a positive linear functional. The independence modulo inner automorphisms of the 1-parameter group is the content of Connes' noncommutative Radon–Nikodym theorem and was extensively exploited by him for the classification of Type III factors. We refer to [4] for a discussion of this material, with an extensive list of references.

I have been interested for many years in trying to understand the classical limit of modular theory, stimulated by two sources. The first is the similarities between KMS theory and symplectic geometry appearing throughout Renault's book [23] (which was also a principal stimulus for the development of the theory of symplectic groupoids). The second is the work of Lichnerowicz and his collaborators as reported in [1]. This interest was revived in recent discussions with Dimitri Shlyakhtenko, who called to my attention the paper [24] in the attempt to interpret some of its contents in terms of symplectic linear algebra. By "Poissonizing" some of the constructions in that paper, I arrived at a definition of the modular automorphism group which is purely geometric.

In fact, I soon discovered that the infinitesimal generator of the modular automorphism group has already appeared in Poisson geometry – it is the curl ("rotationnel" in French) of the Poisson structure. Introduced without a name by Koszul [12], who already noted its modular nature in the case of the dual of a Lie algebra, the curl was named in [7], where it was used in the classification of quadratic Poisson structures (see also [14]). Finally, Brylinski and Zuckerman have recently studied the modular vector field in the context of complex analytic Poisson geometry [3].

For background material on Poisson manifolds and operator algebras, we refer the reader to [30] and [4], respectively.

2. Definition of the modular vector field

Let P be a Poisson manifold with Poisson tensor π, and choose a positive smooth density μ on P. To this data we associate the operator $\phi_\mu : f \mapsto \mathrm{div}_\mu H_f$, where H_f is the hamiltonian vector field of f, and the divergence $\mathrm{div}_\mu \xi$ of a vector field ξ is the function $\mathcal{L}_\xi \mu/\mu$. ($\mathcal{L}_\xi$ is the Lie derivative by ξ.) Although ϕ_μ appears to be a second-order operator (a kind of "laplacian"), a simple computation using the antisymmetry of the Poisson tensor shows that ϕ_μ is in fact a derivation and hence a vector field; we call it the *modular vector field* of (P, π) with respect to the density μ. A further calculation shows that $\mathcal{L}_{\phi_\mu} \mu$ and $\mathcal{L}_{\phi_\mu} \pi$ are both zero. [2]

The modular vector field ϕ_μ is zero precisely when μ is an invariant density for the flows of all hamiltonian vector fields. In this case, we simply refer to μ as an *invariant density* for the Poisson manifold, and we call the Poisson manifold *unimodular*.

[2] See Section 5.

If we replace μ by $a\mu$, where a is a positive function, the modular vector field becomes $\phi_{a\mu} = \phi_\mu + H_{-\log a}$. We conclude that the modular vector field is well-defined modulo hamiltonian vector fields. In other words, the set of modular vector fields for all possible positive densities is an element of the first Poisson cohomology space of P (Poisson vector fields modulo hamiltonian vector fields). We call it the *modular class* of the Poisson manifold. It vanishes just for unimodular Poisson manifolds.

In algebraic terms, the modular class is a derivation of the Poisson algebra $C^\infty(P)$, modulo inner derivations. When we integrate a particular modular vector field (assuming it to be complete), the result is a 1-parameter subgroup in the group $\mathrm{Aut}(C^\infty(P))$ of automorphisms of the Poisson algebra which is intrinsic modulo the subgroup $\mathrm{Aut}_0(C^\infty(P))$ consisting of those automorphisms obtained by integrating time-dependent hamiltonian vector fields. More precisely, the flows of ϕ_μ and $\phi_{a\mu}$ are related by a canonical 1-cocycle on $\mathbb{R}$ with values in the group $\Gamma(P)$ of exact lagrangian bi-sections of the symplectic groupoid of P. (See Section 7, where we will also explain how the elements $\mathrm{Aut}_0(C^\infty(P))$ play the role of inner automorphisms. A hint of this relation is already given in [23]: the group $\Gamma(P)$ is in some sense the classical limit of the unitary group in the multiplier algebra of the algebra whose classical limit is $C^\infty(P)$.)

The modular vector field can also be viewed [12] as the result of applying a differential operator to the Poisson tensor itself. A smooth density μ on P sets up an isomorphism $\alpha \mapsto \alpha \lrcorner \mu$ (defined modulo a local choice of sign, which disappears in what follows) between differential forms and multivector fields on P, so that the exterior derivative becomes an operator on multivector fields. Applying this to the bivector field π yields its modular vector field.

When the modular vector field is zero, the form $\pi \lrcorner \mu$ is closed and therefore defines a deRham cohomology class, which is dual to a homology class in $H_2(P, \mathbb{R})$.[3] As μ runs over all choices of invariant densities, we obtain a convex cone in $H_2(P, \mathbb{R})$ whose elements may be called the *fundamental cycles* of the Poisson structure. If there exists an invariant density of finite total volume, we can restrict attention to invariant densities of total measure 1, in which case we obtain another convex set – the *normalized* fundamental cycles. We discuss these invariants in further detail in Section 9 below.

The modular vector field is also related to the canonical homology of Koszul and Brylinski (see [12,30]), given by the complex in which the chains are differential forms and the boundary operator is $\delta = i_\pi \circ d - d \circ i_\pi$. Suppose that P is oriented, so that we can identify densities with differential forms of top degree. A density μ is thus a top-dimensional chain for Poisson homology. Its boundary $\delta\mu$ is equal to $-\mathrm{d}(\pi \lrcorner \mu) = -\pi \lrcorner \phi_\mu$. Thus, the modular vector field corresponds to the (d and δ exact) $n-1$ form $\delta\mu = -\mathrm{d}(\pi \lrcorner \mu)$. In the unimodular situation, an invariant density μ is a cocycle and thus defines a nonzero element of the top-degree Poisson homology of P. Also note that the zeroth Poisson homology $C^\infty(P)/\{C^\infty(P), C^\infty(P)\}$ is dual to the "traces" on $C^\infty(P)$. Such (smooth) traces are given precisely by the invariant densities. (See Section 3 below.)

[3] Here we must assume P oriented or use twisted coefficients.

This relation between Poisson homology spaces in complementary dimensions is evidently related to Poincaré duality. In fact, such duality (or at least a pairing) has been studied recently by Evens et al. [8], as well as by Brylinski and Xu.

3. Relation to operator algebras

For a von Neumann algebra A, the point of departure for the definition of the modular automorphism group is the choice of a weight: a certain type of positive linear functional on A. The modular automorphism group measures the extent to which the weight fails to be a trace, i.e. to vanish on commutator brackets. According to the noncommutative Radon–Nikodym theorem, any other weight is related to the first by the inner automorphisms associated to a cocycle, with values in the unitary group of A.

Weights on the Poisson algebra of a Poisson manifold P are by definition positive (Borel) measures on P. To fix a measure class (which corresponds to the selection of a particular enveloping von Neumann algebra for an incomplete $*$-algebra), we choose the smooth densities. (A discussion of more general densities, and the corresponding modules obtained by a Poisson version of the GNS construction, is planned for [33].) Now recall that the modular vector field measures the extent to which hamiltonian vector fields are divergence free. Since, for compactly supported g, we have by Stokes' theorem

$$\int_P \{f, g\}\mu = \int_P g\mathcal{L}_{H_f}\mu,$$

it follows that the modular vector field also measures the extent to which integration with respect to μ vanishes on Poisson brackets (with at least one entry compactly supported), i.e. when integration with respect to μ fails to be a "Poisson trace".

The relation

$$\int_P (\{f, g\} - (\phi_{\mu f})g)\mu = 0$$

is called the *infinitesimal KMS condition* relating the weight μ to the vector field ϕ_μ.

4. Some examples

The modular class of a symplectic manifold is zero. In fact, the Liouville density associated to the symplectic structure is invariant under all hamiltonian flows, so the corresponding modular vector field is zero. If we take instead the density $a\mu$, where μ is the Liouville density, we obtain as modular vector field the hamiltonian vector field $H_{-\log a}$. Writing E for the hamiltonian $-\log a$, we find a natural association between the hamiltonian flow of the function E and the density $e^{-E}\mu$. This association, familiar in classical statistical mechanics, was the starting point for the development of the KMS theory in quantum statistical mechanics, which was subsequently shown to be essentially equivalent to Tomita–Takesaki theory. (A geometric interpretation of the KMS theory in terms of conformal deformations of Poisson brackets and their quantizations was given in [1].)

If P is the dual of a Lie algebra $\mathfrak{g}$, with its Lie–Poisson structure, the modular vector field with respect to any translation-invariant density is the constant vector field with value trad, the trace of the adjoint representation, or *modular character* of the Lie algebra. (This was already observed in [12].) In particular, the modular class of $\mathfrak{g}^*$ is zero just when $\mathfrak{g}$ is unimodular, since a hamiltonian vector field must vanish at the origin in any Lie–Poisson structure. This fact motivates our use of the term unimodular to describe Poisson structures with zero modular class. (Such structures were called "exact" in [14].)

At any singular point of a Poisson manifold, the projection of the modular vector field into the normal space of the symplectic leaf is equal to the modular character of the transverse Lie algebra. Consequently, the transverse Poisson structure (and hence the Poisson structure itself, at least locally) admits a 1-parameter group of symmetries in this direction, even if the transverse structure is not linearizable. (See [31,32] for discussion and examples of nonlinearizable structures.)

If P is 2-dimensional, with Poisson structure given in coordinates by $\{x, y\} = f(x, y)$, then the modular vector field with respect to $\mu = |\, \mathrm{d}x \wedge \mathrm{d}y|$ is the same as the hamiltonian vector field for f *with respect to the canonical bracket* $\{x, y\} = 1$. In particular, the modular vector field is tangent to the zero level of f, which is the singular set of the Poisson structure, and the restriction of the modular vector field to this singular set is invariantly attached to the Poisson structure.

For the Lie–Poisson structure on $\mathbb{R}^2$ with defining relation $\{x, y\} = y$, the modular flow with respect to translation-invariant measures is given by translations in the x-direction. The upper half plane H^+ in this space is a symplectic manifold. The smooth measures on H^+ with smooth continuations to $\mathbb{R}^2$ could be thought of as a measure class on H^+ with distinguished behavior "at infinity", giving rise to a nontrivial modular flow at infinity.

The Poisson manifolds with boundary which are locally equivalent to the product of the upper half plane with symplectic manifolds are exactly the b-symplectic manifolds of Nest and Tsygan [22], arising by generalization from the b-cotangent bundles of manifolds with boundary studied by Melrose [19]. The modular vector field on the boundary gives the obstruction to the existence of a trace on algebras of b-pseudodifferential operators, or more generally on quantized Poisson algebras of b-symplectic manifolds.

As was noted and exploited in [7,14], the modular vector field of a quadratic Poisson structure on a vector space (with respect to translation-invariant measures) is a linear vector field (which is an invariant of the modular class). For instance, the modular flow for the structure $\{x, y\} = x^2 + y^2$ consists of rotations around the origin. Again, we could consider this as the modular flow at "infinity" (here represented by the origin) for the symplectic structure on the punctured plane, with boundary conditions determined by extendibility over the puncture. This Poisson structure is also the local model for the singularity of the Bruhat–Poisson structure on S^2. (See [16].)

The modular vector field has also been calculated in [8] for the Bruhat–Poisson structures on higher-dimensional flag manifolds, as well as for the related compact Poisson Lie groups. The nonvanishing of the modular class for these manifolds seems to be related to the fact that the "Haar measure" on the corresponding quantum groups is not a trace, but rather satisfies a KMS-type condition in which the modular automorphism is related to the square of the antipode. The recent paper [18] contains an extensive discussion of quantum groups from the von Neumann algebra point of view; a study of this paper from the Poisson algebra point of view should give some new insight both into quantum groups and their classical limit. It would also be interesting to interpret the modular vector field on the flag manifold in terms of the geometry at infinity of the "big cell" (an open dense symplectic leaf).

We close this section with a possible application of the modular class. Tuynman [29] has pointed out a correction which should be made to geometric quantization in order to make it compatible under reduction by a nonunimodular group G acting on a symplectic manifold P. It seems that there should be an interpretation of his results in terms of the nonunimodularity of the Poisson manifold $\mathfrak{g}^*$ and that of P/G, which is Morita equivalent to $\mathfrak{g}^*$ when the G action is free. In fact, the first Poisson cohomology spaces of Morita equivalent Poisson manifolds are isomorphic, according to Ginzburg and Lu [10], and Ginzburg [9] has shown that the modular classes are compatible with this isomorphism.

5. Regular Poisson manifolds

Near any regular point x on a Poisson manifold, we can introduce canonical local coordinates and hence a measure which, near x, is invariant under all hamiltonian flows. The modular vector field with respect to this measure is therefore zero near x. Since x is arbitrary, the modular vector field with respect to *any* measure is locally hamiltonian throughout the set of regular points of P.[4] In particular, any modular vector field is tangent to all the regular symplectic leaves.[5]

On the other hand, global conditions can cause the modular class of a regular Poisson manifold to be nonzero. For example, we consider the regular Poisson structures on $\mathbb{R}^2 \times S^1$, with coordinates (x, y, θ), of the form

$$\pi = \frac{\partial}{\partial y} \wedge \left(\frac{\partial}{\partial \theta} + g(x) \frac{\partial}{\partial x} \right),$$

where $g(x) = 0$ just at $x = 0$. The symplectic leaves for this structure consist of the cylinder C defined by $x = 0$ and a family of planes which spiral around this cylinder.

For $\mu = \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y$, we have $\pi \lrcorner \mu = \mathrm{d}x + g(x)\,\mathrm{d}\theta$, $\mathrm{d}(\pi \lrcorner \mu) = g'(x)\,\mathrm{d}x \wedge \mathrm{d}\theta$, and hence $\phi_\mu = -g'(x)(\partial/\partial y)$. If $g'(0) \neq 0$, the restriction of ϕ_μ to the cylinder C is a nonzero multiple of $\partial/\partial y$, which is not hamiltonian, so the modular class of this Poisson structure is nonzero. Perhaps more surprising is that, if $g'(0) = 0$, ϕ_μ is still not hamiltonian, even

[4] In [3], the modular vector field is considered as a section of a sheaf of Poisson modulo hamiltonian vector fields, so that it is actually supported on the set of singular points. An advantage of this framework is that the modular vector field is defined even when there is no global volume element, a situation which often occurs in the holomorphic setting.

[5] Since the regular points of any Poisson manifold form a dense subset, this argument also gives a quick proof that $\mathcal{L}_{\phi_\mu} \mu$ and $\mathcal{L}_{\phi_\mu} \pi$ are zero on all of P.

though its restriction to each symplectic leaf is hamiltonian. (It is zero on C, and each of the remaining leaves is simply connected.) To see this, we look at the most general hamiltonian vector field

$$H_f = \tilde{\pi}(\mathrm{d}f) = \frac{\partial f}{\partial y}\left(\frac{\partial}{\partial \theta} + g(x)\frac{\partial}{\partial x}\right) - \left(\frac{\partial f}{\partial \theta} + g(x)\frac{\partial f}{\partial x}\right)\frac{\partial}{\partial y}.$$

If H_f is to equal ϕ_μ, we must have

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial \theta} + g(x)\frac{\partial f}{\partial x} = g'(x).$$

To see that this is impossible, we introduce the averaged hamiltonian function $F(x) = \int_0^{2\pi} f(x, y, \theta)\,\mathrm{d}\theta$ (independent of y by the first equation above), and we integrate the second equation above with respect to θ to get

$$g(x)F'(x) = g'(x).$$

For $x \neq 0$, we have $F(x) = \ln|g(x)| + C$, where C is constant on each semiaxis. As $x \to 0$, $g(x) \to 0$ implies that $F(x) \to \infty$, which is impossible if f is continuous.

The modular vector field of a regular Poisson manifold is closely related to an object which depends only on the foliation by symplectic leaves. If $\mathcal{F}$ is a foliation on P with tangent bundle $F \subset TP$, we can define its modular class in the following way. Choose a smooth transverse positive density ν for $\mathcal{F}$; i.e. a nowhere-vanishing section of the highest exterior power of the normal bundle TP/F (which for simplicity we assume to be oriented – otherwise the usual twisting by an orientation bundle is needed). If we denote by ∇ the Bott connection of the foliation (extended to densities), then $\nabla\nu/\nu$ is a well-defined closed (because the Bott connection is flat) 1-form along the leaves of $\mathcal{F}$, which we will denote by ψ_ν. The integral of ψ_ν around a loop in a leaf of $\mathcal{F}$ is the logarithm of the determinant of the linearized holonomy of the loop, and form ψ_ν is zero exactly when ν defines an invariant transverse measure to the foliation. If we multiply ν by a positive function a, ψ_ν changes to $\psi_\nu + d_{\mathcal{F}}(\ln a)$, so that $[\psi_\nu]$ is a well-defined class in the tangential cohomology of the foliation $\mathcal{F}$, which we call the *modular class of the foliation*.

On any regular Poisson manifold, multiplication by the canonical symplectic density along the leaves gives a 1–1 correspondence between the transverse densities to the symplectic leaf foliation and the densities on the ambient manifold. Given a transverse density ν, we can apply the Poisson tensor to the closed form ψ_ν along the leaves to obtain a locally hamiltonian vector field tangent to the leaves which is precisely the modular vector field ϕ_μ associated to the density μ corresponding to ν.

When the symplectic leaf foliation is co-oriented of codimension 1, a transverse density is given by a 1-form λ which annihilates the leaves. Integrability means that $\mathrm{d}\lambda = \alpha \wedge \lambda$ for a 1-form α defined up to a multiple of λ. The restriction of α to the leaves depends only on λ and is in fact ψ_λ. When the Poisson structure is unimodular, the form λ can be chosen to be closed, so that $\alpha \wedge \mathrm{d}\alpha$, which represents the *Godbillon–Vey class* of the foliation, is zero. Hence a nonvanishing Godbillon–Vey class implies nonunimodularity. The much stronger, but more difficult to prove, theorem of Hurder–Katok cited in [4, p. 261], establishes that the nonvanishing of the Godbillon–Vey class implies the nonexistence of any invariant transverse measure, smooth or not.

Example. On the group $PSL(2,\mathbb{R})$ we have the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of left-invariant vector fields, satisfying the commutation relations

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \qquad [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_1, \qquad [\mathbf{e}_3, \mathbf{e}_2] = -\mathbf{e}_2.$$

These pass to any quotient $PSL(2,\mathbb{R})/\Gamma$ by a discrete subgroup Γ acting from the left, as does the dual basis $(\omega_1, \omega_2, \omega_3)$. We consider the Poisson structure $\pi = \mathbf{e}_2 \wedge \mathbf{e}_3$ and the density $\mu = \omega_1 \wedge \omega_2 \wedge \omega_3$, with respect to which the modular vector field is $-e_2$. If Γ is the fundamental group of a compact Riemann surface M, then $PSL(2,\mathbb{R})$ is the unit tangent bundle of M, $\mathbf{e}_3$ is the geodesic flow for the Poincaré metric, and the foliation by symplectic leaves of π is the foliation by stable manifolds. The symplectic form along the leaves is the area form for the natural induced metric on the tangent bundle, and the modular vector field is the generator of the horocycle flow! Since e_1 is the direction of the unstable manifolds, if we travel around a closed geodesic, the integral of the modular 1-form (which is ω_3) must be nonzero, so the modular class is nontrivial.

We can easily compute the Godbillon–Vey class in this example. Taking ω_1 as our transverse density λ, we have $\mathrm{d}\omega_1 = \omega_1 \wedge \omega_3$, so $\alpha = -\omega_3$, and $\alpha \wedge \mathrm{d}\alpha = \omega_3 \wedge \mathrm{d}\omega_3 = -\omega_3 \wedge \omega_1 \wedge \omega_2$, a volume form which gives a nonzero Godbillon–Vey class.

It is interesting to compare our discussion with that of [4, p. 58]. It is stated there that when the stable foliation is "suspended" to the bundle of transverse densities, the resulting foliation of type II has the same space of leaves as the horocycle flow. We give a geometric explanation for this occurrence of the horocycle flow in Section 8.

The Reeb foliation on S^3 provides another instructive example. Its modular class is nonzero, since the contractive nature of the holonomy around the central torus precludes the existence of a smooth positive invariant transverse measure. However, the linearized holonomy is zero around the torus, so the modular vector field can be zero there. Since all the other leaves are planes, the modular vector field is hamiltonian on every leaf separately, but it is not globally hamiltonian. On the other hand, in this case, the Godbillon–Vey class is zero. (See for example [28].)

The situation for regular Poisson manifolds suggests a point of view for an arbitrary Poisson manifold P. A density on P should be thought of as a way of "encoding" a transverse measure to the foliation by symplectic leaves, and the modular vector field measures the extent to which this transverse measure fails to be "invariant under holonomy". This point of view may shed some light on the problem of defining the notion of transverse structure, holonomy, and modular classes for singular foliations. (Compare [6] and [26].) Some further insight in this direction may be found in our discussion of Lie algebroids in Section 7.

Given any foliation of a manifold M, the cotangent bundle P along the leaves is a Poisson manifold in a natural way. A transverse density to the first foliation pulls back to a transverse density to the symplectic foliation of P, and the modular 1-form pulls back accordingly. Thus, the modular vector field of the cotangent bundle to a foliation is generated by the pullback of the modular 1-form of the foliation. In this way, the modular construction for Poisson manifolds is a generalization, as well as a special case (!) of the construction for foliations.

Finally, we note that Mikami [20] has also studied the Godbillon–Vey class in the context of Poisson geometry.

6. Intrinsic completeness

The modular flow of an operator algebra is always an action of the real numbers. For a Poisson manifold, the modular vector field defines an action of the real numbers only when it is complete. For a given Poisson structure, this completeness may depend on which representative of the modular class is chosen. (For instance, any hamiltonian vector field on a symplectic manifold is a modular vector field.) We will call a Poisson manifold *intrinsically complete* if *some* representative of the modular class is a complete vector field.

Intrinsic completeness is clearly an invariant property of a Poisson manifold. It would be nice to have a striking characterization of this property, and some interpretation of its meaning (perhaps in connection with some form of quantization), but we must content ourselves here with some examples.

An open subset of $\mathbb{R}^2$ with the Poisson structure $\{x, y\} = y$ is intrinsically complete if and only if it contains either all or none of the x-axis, since every modular vector field is constant and nonzero on that axis. Of course, every symplectic manifold is intrinsically complete. On the other hand, even a regular Poisson manifold can fail to be intrinsically complete. For instance, although the Poisson structures on $\mathbb{R}^2 \times S^1$ described in the previous section are intrinsically complete, the restriction to the region $|y| < 1$ is not intrinsically complete when $g'(0) \neq 0$. (It is not clear what happens when $g'(0) = 0$.) This follows from the following lemma.

Lemma 6.1. *On the cylinder $(-1, 1) \times \mathbb{R}$ with symplectic structure $\mathrm{d}y \wedge \mathrm{d}\theta$, every complete, locally hamiltonian vector field is globally hamiltonian.*

Proof. Let X be a locally hamiltonian vector field, $\alpha = X \lrcorner (\mathrm{d}y \wedge \mathrm{d}\theta)$. The integral of α around a loop encircling the cylinder gives the flux of area through the loop under the flow of X. Since the cylinder has finite area, this flux must be zero, so α is exact, and X is globally hamiltonian. □

7. Modular classes of Lie algebroids and groupoids

The foliation and Lie–Poisson examples are special cases of a more general construction. Given a Lie algebroid $\mathcal{A}$, the dual bundle carries a natural Poisson structure. The modular vector field of this structure with respect to a suitably chosen density is tangent to the fibers of $\mathcal{A}^*$ and translation invariant along each fibre. Thus this vector field can be identified with a section of $\mathcal{A}^*$. As such, it is a 1-cochain for the Lie algebroid cohomology [17] of $\mathcal{A}$; it turns out to be a cocycle whose cohomology class is well defined. We have therefore attached to each Lie algebroid a *modular class* in its first Lie algebroid cohomology.

In fact, this class can be defined directly in terms of the Lie algebroid itself. We summarize here the detailed discussion which can be found in [8].

Given a Lie algebroid $\mathcal{A}$ over P, we introduce the bundle $Q_{\mathcal{A}} = \wedge^{\text{top}}\mathcal{A} \otimes \wedge^{\text{top}} T^*P$. When $\mathcal{A}$ is an integrable subbundle of TP, this is the bundle whose sections are (not necessarily invariant or positive) transverse smooth measures to the corresponding foliation, so we should think of sections of $Q_{\mathcal{A}}$ in general as being "transverse measures to $\mathcal{A}$."

It turns out that the "difference" between Lie derivative operators on $\wedge^{\text{top}}\mathcal{A}^*$ and $\wedge^{\text{top}}T^*P$ defines a representation of $\mathcal{A}$ on the line bundle $Q_{\mathcal{A}}$. Any nowhere vanishing section of this line bundle (or its square, in case the "transverse space is nonorientable") has a "divergence" which gives an invariantly defined class $\theta_{\mathcal{A}}$, the *modular class* of $\mathcal{A}$, in the Lie algebroid cohomology of $\mathcal{A}$ (with coefficients in the trivial line bundle). This construction reproduces the class defined above using the Poisson structure on $\mathcal{A}^*$.

Since the cotangent bundle of a Poisson manifold is a Lie algebroid, we can consider the modular class of this Lie algebroid. But the Lie algebroid cohomology of T^*P is just the Poisson cohomology of P, and in fact we arrive back at the modular class of the Poisson manifold P itself, if to each density μ on P we construct a section ρ_μ of Q_{TP} by first taking the "square" of μ as a section of Q_{TP} and then taking its square root in the same line bundle.

We turn now to Lie groupoids. Once again, we give here a brief description of results to be presented in more detail in [8]. The modular class of a Lie algebroid is in fact the infinitesimalization of the modular class of a corresponding (local) Lie groupoid, which may be defined as a ratio of right- and left-invariant "measures," as in [23, Chap. 1, Section 3]. For a Lie groupoid with Lie algebroid $\mathcal{A}$ we can give a description of the modular class which does not require the separate choice of a Haar system and a measure on the base. Instead, we choose a section of $Q_{\mathcal{A}}$, which plays the role of these two objects simultaneously. In fact, it can be shown that G acts in a natural way on the line bundle $Q_{\mathcal{A}}$, so that a trivialization of $Q_{\mathcal{A}}$ turns this action into a 1-cocycle on G, with values in the multiplicative group $\mathbb{R}^\times$ of the real numbers; this cocycle is called the *modular function* of the groupoid with respect to the given section. [6] A change of trivialization corresponds to a zero cocycle, whose coboundary gives the change in the modular function, so that the *modular class* is a well-defined element of $H^1(G; \mathbb{R}^\times)$. Differentiation of this modular class (more precisely, of its representative cocycles) along the identities of G gives the modular class of the Lie algebroid $\mathcal{A}$.

When $\mathcal{A}$ is the cotangent bundle of a Poisson manifold P, then the groupoid G can be taken to be a symplectic groupoid for P, if it exists. [7] This gives the following picture. Given a density μ on P, its modular flow [8] lifts in a natural way to a flow on the symplectic groupoid

[6] Multiplication by powers of this function gives the modular automorphism group of the von Neumann algebra of the groupoid; see [23, p. 115].

[7] See [5] or [30] for a discussion of symplectic groupoids, and [34] for the lifting of Poisson automorphisms to groupoid automorphisms.

[8] All flows in this paragraph will be local, if necessary.

G by symplectic groupoid automorphisms. This lifted flow is hamiltonian and is generated by the logarithm of the modular function f on G. When f is the coboundary of a function h on P, the logarithm of h is a hamiltonian for the modular flow on P. In this situation, we can now give a precise sense in which the modular flow is "inner". (Compare [23, p. 111].) Namely, the pullback of f to G by the source map of this groupoid itself generates a hamiltonian flow on G which moves the identity section into a 1-parameter subgroup of the group of $\Gamma(P)$ of exact lagrangian bi-sections in G. The action of this 1-parameter subgroup by conjugation on G is exactly the lifted modular flow, and its restricted action on the identity section is the modular flow itself. But the group $\Gamma(P)$ should be thought of as being essentially a classical limit of the unitary elements in the (multiplier algebra of) "quantized" algebra whose classical limit is the functions on P. (The Lie algebra of $\Gamma(P)$ consists of the real-valued functions on P, which is the classical limit of the self-adjoint elements of the quantized algebra.)

We note here that an $\mathbb{R}^\times$-valued 1-cocycle ϕ on the symplectic groupoid G gives rise to a 1-parameter group of automorphisms of the groupoid algebra of G in two different ways: first by multiplication by the powers ϕ^{it} (see [23]), and second by the hamiltonian flow of $\log|\phi|$. The relation between these two groups of automorphisms is not completely clear to us, but it must somehow involve the passage, via quantization as in [34], from the groupoid algebra of G to the quantized algebra of functions on the underlying Poisson manifold P.

We also remark that the inversion map in the symplectic groupoid G should be the canonical transformation underlying the $*$-operator in the quantized algebra of functions on P, but there may also need to be an "amplitude" as in the theory of Fourier integral operators. We suspect that this amplitude is related to the modular function via the operator-algebraic construction of the latter, as described for instance in [24]. We also suspect a relation with the square of the antipode in the "hopfoid algebra" of functions on the groupoid, which should lead us back to the ideas about quantum groups mentioned at the end of Section 4.

8. The flow of weights

In the theory of von Neumann algebras, the modular flow leads to two ways of passing between algebras of type II and type III. In this section, we discuss the Poisson analogs of those constructions.

Given a Poisson manifold P with positive density μ and corresponding modular vector field ϕ_μ, we can construct the Poisson semidirect product of P by this modular flow (even if it is not complete). This Poisson analog of the crossed product of an algebra by a 1-parameter group of automorphisms is defined as follows.

Given a Poisson vector field ϕ on a Poisson manifold P, we define the *semidirect product*[9] to be the quotient of $P \times T^*\mathbb{R}$ by the diagonal action of $\mathbb{R}$, which acts on its own cotangent bundle by left translations. Although this construction appears to depend on the

[9] This construction may be found in [13, Section 2], or the appendix of [32]. The name comes from the fact that, when P is the dual of the Lie algebra on which a group acts by automorphisms, the construction produces the dual of the semidirect product Lie algebra.

integration of ϕ to a flow, we can identify the quotient with $P \times \mathbb{R}^*$, on which the induced Poisson structure is given purely in terms of the Lie algebra action by the requirement that the projections on P and $\mathbb{R}^*$ be Poisson maps, and that $\{f, \tau\} = \phi \cdot f$, where f is any function on P and τ is the standard coordinate on $\mathbb{R}^*$. (Our notation has been chosen to suggest how this construction can be extended to the case where $\mathbb{R}$ is replaced by an arbitrary Lie algebra acting on P by Poisson vector fields.) We denote $P \times \mathbb{R}^*$ with the resulting Poisson structure by $P \times_\phi \mathbb{R}^*$. If the vector field ϕ is globally hamiltonian, then the choice of a hamiltonian function produces an isomorphism between $P \times_\phi \mathbb{R}^*$ and the direct product $P \times \mathbb{R}^*$ (see [13]). More generally, the Poisson isomorphism class of the semidirect product depends on only the equivalence class of ϕ modulo globally hamiltonian vector fields.

Applying the semidirect product construction to the modular flow, we obtain a Poisson manifold $P \times_{\phi_\mu} \mathbb{R}^*$ which is determined up to isomorphism by the Poisson manifold P itself. (This is the Poisson counterpart of a result in [27], where it is established that the cross product by the modular flow is weight-independent.) It turns out that $P \times_{\phi_\mu} \mathbb{R}^*$ is a unimodular Poisson manifold; in fact, it is simple to check that $\mu \wedge e^\tau \, d\tau$ is an invariant density.

The *flow of weights*, denoted by $mod(P)$, is defined to be the flow of the vector field $\partial/\partial\tau$ on the space of leaves of $P \times_{\phi_\mu} \mathbb{R}^*$.

In case P is a regular Poisson manifold, the objects introduced above have a rather simple description. The symplectic leaves of $P \times_{\phi_\mu} \mathbb{R}^*$ are coverings of the symplectic leaves of P; in fact, they are the parallel (multivalued) sections for a flat connection along the symplectic leaves of P on the bundle $P \times \mathbb{R}^*$. Upon identification of $P \times \mathbb{R}^*$ with the space of transverse densities to the symplectic leaf foliation (via exponentiation and multiplication by the transverse density associated with μ), this connection is just the usual Bott connection. This gives a completely intrinsic description (corresponding to the "functorial construction" on p. 496 of [4]) of the semidirect product manifold $P \times_{\phi_\mu} \mathbb{R}^*$ and hence of the flow of weights: the flow is given by the action of $\mathbb{R}^+$ by multiplication on the "space of multivalued invariant transverse densities" to the symplectic leaf foliation.

Note that the flow of weights, unlike the modular flow itself, depends on only the symplectic leaf foliation. In fact, it corresponds precisely to the flow of weights for a foliation algebra as described in Proposition 9c on p. 58 of [4]. We also remark that any group of Poisson automorphisms of P acts naturally on the flow of weights.

The second way to pass from a general Poisson manifold to a unimodular one is to divide by the modular flow. Of course, the resulting quotient, which we denote by P/ϕ_μ, is a manifold, even locally, only if the modular vector field is either identically zero or nowhere zero. In the latter case, it can be shown at least formally (i.e. without worrying about the global structure of quotient spaces) that the semidirect product $P \times_{\phi_\mu} \mathbb{R}^*$ is Morita equivalent as a Poisson manifold [35] to the quotient space. (Locally, the semidirect product is the product of the quotient space by a 2-dimensional symplectic manifold.)

We can show directly that the leaf spaces of $P \times_{\phi_\mu} \mathbb{R}^*$ and P/ϕ_μ are isomorphic when P is regular and ϕ_μ is nowhere zero. To do this, we restrict attention to one symplectic leaf $\mathcal{O}$ of P at a time.

Let $F \subset T\mathcal{O}$ be the symplectic orthogonal space to ϕ_μ; it is also the null bundle of the closed 1-form ψ_μ corresponding to ϕ_μ via the symplectic structure on $\mathcal{O}$. Choose a vector field X on $\mathcal{O}$ such that $\psi_\mu(X) = 1$, and let $\tilde{X}$ be its horizontal lift to $\mathcal{O} \times \mathbb{R}^*$ for the "Bott connection". $\tilde{X}$ is a complete vector field, and $d\tau(\tilde{X}) = 1$, so the coordinate function τ is a fibration to $\mathbb{R}^*$ on each leaf of $\mathcal{O} \times_{\phi_\mu} \mathbb{R}^*$. Hence, the intersection of each such leaf with $\tau^{-1}(0)$ is connected and must be a leaf of the foliation $\mathcal{F}$ determined by F. Thus, the symplectic leaf space of $\mathcal{O} \times_{\phi_\mu} \mathbb{R}^*$ is isomorphic to the leaf space of $\mathcal{F}$. On the other hand, it is a basic fact about Poisson reduction that the symplectic leaf space of $\mathcal{O}/\phi_\mu$ is the same as the leaf space of $\mathcal{F}$.

In particular, the symplectic leaf space for $P \times_{\phi_\mu} \mathbb{R}^*$ in the example of $PSL(2, \mathbb{R})/\Gamma$ in Section 5 is the same as the space of leaves of the horocycle foliation. (When the symplectic leaves of P are 2-dimensional, the bundle F is the same as the span of ϕ_μ.) This corresponds to a similar statement about the horocycle foliation on p. 58 of [4].

To close this section, we propose the study of the Poisson analog of the invariants R and S of von Neumann algebras discussed in [4].

9. The fundamental class of a unimodular Poisson manifold

Let (P, π) be a unimodular Poisson manifold of dimension n. For simplicity, we will assume that P is oriented.[10] Beginning with an invariant positive density μ on P we obtain in succession the closed $n-2$-form $\pi \lrcorner \mu$, the de Rham cohomology class $[\pi \lrcorner \mu] \in H^{n-2}(P; \mathbb{R})$, and the dual homology class $\sigma_\pi(\mu) \in H_2(P; \mathbb{R})$. When P is noncompact, we use homology with locally finite chains to insure Poincaré duality.

Let $I(P, \pi)$ be the convex cone of invariant positive densities, and $I_1(P, \pi)$ its convex subset of normalized elements (i.e. satisfying the condition $\int_P \mu = 1$). We say that (P, π) has *finite type* when $I_1(P, \pi)$ is nonempty. The images of $I(P, \pi)$ and $I_1(P, \pi)$ under σ_π are a convex cone $\mathcal{C}(P, \pi)$and a convex set $\mathcal{C}_1(P, \pi)$, respectively, in $H_2(P; \mathbb{R})$ whose elements we call the *fundamental cycles* and *normalized fundamental cycles* of the unimodular Poisson manifold.

If (P, π) is a connected symplectic manifold of dimension $n = 2m$ with symplectic form ω, oriented by ω^m, then $I(P, \pi) = \{c\omega^m \mid c > 0\}$. If P has finite symplectic volume, then it is of finite type, and $I_1(P, \pi) = \{\omega^m / v(M)\}$, where $v(P) = \int_P \omega^m$, which is $n!$ times the symplectic volume. Since $\pi \lrcorner \omega^m = \omega^{m-1}$, if we denote by u_ω the homology class dual to ω^{m-1}, then $\mathcal{C}(P, \pi)$ is the ray through u_ω, and $\mathcal{C}_1(P, \pi)$ consists of the single class $u_\omega / v(P)$. Note that the pairing $\langle [\omega], u_\omega \rangle$ is by definition $\int_P \omega \wedge \omega^{m-1} = v(P)$, so the homology class in $\mathcal{C}_1(P, \pi)$ is normalized so that its pairing with the symplectic class $[\omega]$ equals 1.

Next suppose that (P, ω) is a bundle of $2k$-dimensional connected symplectic manifolds; i.e. its symplectic leaves are the fibers of a smooth fibration $\gamma : P \to M$. Let ω be a 2-form

[10] Everything we will do also works in the nonorientable case if we use forms, homology, and cohomology with twisted coefficients.

on P (not necessarily closed) which restricts to the given symplectic structure $\omega(y)$ on each leaf $P(y) = \gamma^{-1}(y)$. The invariant densities on P are then of the form $\omega^k \wedge \gamma^*\nu$, where ν runs over the volume elements on M consistent with the orientation of M corresponding to a given orientation of P. (This follows from the corresponding fact about the densities on an exact sequence of vector spaces.) The corresponding closed form $\pi \lrcorner (\omega^k \wedge \gamma^*\nu)$ equals $\omega^{k-1} \wedge \gamma^*\nu$.

In terms of the homology maps $i(y) : H_2(P(y); \mathbb{R}) \to H_2(P; \mathbb{R})$ induced by the inclusions, the homology class dual to $\pi \lrcorner (\omega^k \wedge \gamma^*\nu)$ is then $\int_P i(y) u_{\omega(y)} \nu$, so that $\mathcal{C}(P, \pi)$ consists of the superpositions (with strictly positive weights) of fundamental cycles of the symplectic leaves, inserted into $H_2(P; \mathbb{R})$ by the inclusion maps. (These inserted homology classes are defined because the inclusions of the fibres are proper maps.)

Remark. Suppose that $\pi = 0$. Then any μ is invariant, but $\pi \lrcorner \mu$ is always zero, so that $\mathcal{C}(P, 0) = \{0\}$. This is consistent with the fact that each symplectic leaf has $H_2 = \{0\}$.

To determine the normalized fundamental cycles on a bundle of symplectic manifolds, we first note that $\int_P \omega^k \wedge \gamma^*\nu = \int_M v(P(y))\nu$, so that (P, π) has finite type when the symplectic volume function of the leaves is locally L^1 on M. In particular, $v(P(y))$ must be finite for almost all y. (Note that the class of locally L^1 functions is determined by the smooth structure of M. It is independent of ν, which can always be chosen to make a given locally L^1 function have integral equal to 1.)

Now for ν such that $\omega^k \wedge \gamma^*\nu$ is normalized we have

$$\int_M i(y) u_{\omega(y)} \nu = \int_M i(y) \frac{u_\omega(y)}{v(P(y))} v(P(y))\nu,$$

which is the integral over M of the images in $H_2(P; \mathbb{R})$ of the normalized fundamental cycles of the symplectic leaves, with respect to the normalized measure $v(P(y))\nu$. In other words, the set of normalized fundamental cycles of P is the "open convex hull" of the inserted normalized cycles of the leaves.

If (P, ω) is a finite union of connected submanifolds, possibly of different dimensions, then the fundamental cycles are again given by taking convex combinations of the fundamental cycles of the components.

The examples above suggest that the fundamental cycles for any unimodular Poisson manifold should be thought of as some kind of superpositions of fundamental cycles of the symplectic leaves. The next example will give some meaning to this viewpoint in the case where P is compact, but the symplectic leaves are not of finite type.

Let P be a 3-torus with a translation invariant Poisson structure π whose symplectic leaves form a foliation by planes, each of which is dense in the torus. In terms of coordinates (x_1, x_2, x_3) (defined modulo $\mathbb{Z}$), we can write

$$\pi = \left(\frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_3} \right) \wedge \left(\frac{\partial}{\partial x_2} + a_2 \frac{\partial}{\partial x_3} \right),$$

where a_2 and a_3 are irrational and have irrational ratio. The density $\mu = b\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$ is invariant only when b is constant and normalized when $b = 1$; in fact, $\pi \lrcorner \mu = b(\mathrm{d}x_3 - a_1\,\mathrm{d}x_1 - a_2\,\mathrm{d}x_2)$, which is closed just when b is constant along the leaves of the foliation. If we denote by c_{ij} the fundamental homology class of the oriented product of the ith and jth coordinate circles, the the homology class dual to $(\mathrm{d}x_3 - a_1\,\mathrm{d}x_1 - a_2\,\mathrm{d}x_2)$ is $c_{12} - a_1 c_{23} + a_2 c_{13}$. This is the unique normalized fundamental cycle, which generates the ray of fundamental cycles. It should be thought of as an inserted fundamental class of any leaf.

The most general Poisson structure having the same symplectic leaf foliation is of the form $c\pi$ for some nowhere vanishing function c. The invariant densities now have the form $b|c^{-1}|\mu$ for positive b, and the ray of fundamental cycles remains the same (up to the sign of c), while the normalized fundamental cycle is multiplied by $(\int_P c^{-1}\mu)^{-1}$. This homology class is therefore an invariant of the Poisson structure; we refer to [11] for a proof that this is essentially the only invariant when the symplectic leaf foliation is translation invariant.

We end this section with a question and a remark. When does the set of normalized fundamental cycles have a compact closure? It seems that these cycles might be related to Ruelle–Sullivan currents for foliations [25] and related currents for group actions studied by Brylinski [2].

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